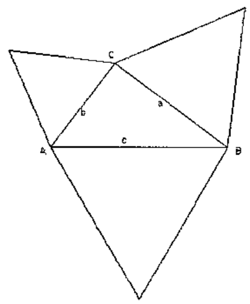
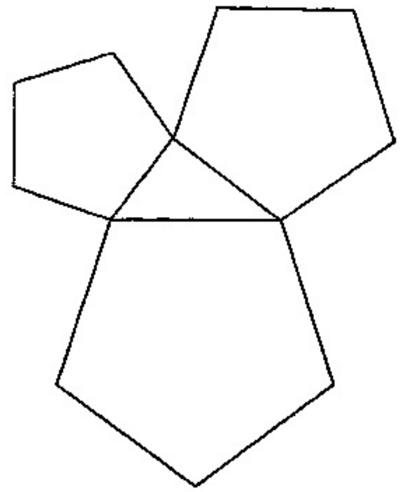
Fun With Pythagoras

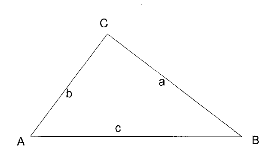




It was established even before Pythagoras' time that a relation similar to that for the squares on the sides of a right triangle existed for all regular polygons. This follows from the fact that the area of a regular polygon of n sides of length s can be expressed as $B_n s^2$. where B_n is a number depending only on n, the number of sides of the regular polygon. A little straightforward trigonometry establishes that $B_n = (n/4)(\cot(180/n))$ with angles in degrees. And the area then is given by:
$A_n = B_n s^2$ in each case.

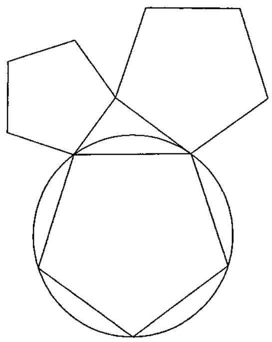
A little punching of keys of a pocket calculator establishes:

$B_3 = 0.433012702...$
$B_4 = 1$ (of course)
$B_5 = 1.720477401...$ etc.



Stating the Pythagoras Theorem in usual notation, $c^2 = a^2 + b^2$ and multiplying through by B_n we have $B_n c^2 = B_n a^2 + B_n b^2$ or, in words:

The area of a regular polygon formed on the hypotenuse of a right triangle equals the sum of the regular polygons of the same number of sides formed on the each of the other two sides.



While it has little direct connection with the Pythagoras Theorem, an application of the above formulae comes up in considering a regular polygon's circumscribed circle, that is, the circle containing the vertices of the polygon.

A little bit of trig establishes that the radius of such a circle is given by $r = s/(2 \sin(180/n))$ or the area, $A_n = \pi s^2/(4 \sin^2(180/n))$. As a bit of a test of these formulae we compare the area of a regular polygon and that of its circumscribed circle. Clearly the area of the polygon is less than that of the circumscribed circle.

For the example pictured (pentagons) the polygon area is
$1.72048\ s^2$ and for the circle is
$2.27328\ s^2$.

But as n increases the polygon area gets closer to that of the circle. If we use our pocket calculator for , say, n = 1,000 we get
$(79577.20975...)\ s^2$ for the polygon and
$(79577.73335...)\ s^2$ for the circle. Close indeed.

(And, if drawn on a five inch hypotenuse the figures would be over 130 feet across.)

Generating Pythagorean Triples

If u and v are relatively prime (highest common divisor = 1) and u > v and

$a = u^2 - v^2$

$b = 2uv$

$c = u^2 + v^2$

Then a, b, and c form a Pythagorean triple, namely $c^2 = a^2 + b^2$

Examples:

u	v	a	b	c
2	1	3	4	5
3	1	8	6	10
3	2	5	12	13
4	3	7	24	25
1001	1000	2001	2002000	2002001
1001	500	752001	1001000	1252001
52	23	2175	2392	2322

Get out your pocket calculator and confirm these results.

Another expansion of Pythagoras

The Pythagorean Theorem is a special case of a general law of triangles which Kaplan hints was known to the pre-Greek mathematicians but couched in very different language. This rule applies to all triangles and is usually know as the Law of Cosines.

In a triangle, ABC with sides a, b, and c opposite the corresponding lettered angles the Law of Cosines can be written:

$c^2 = a^2 + b^2 - 2abCosA$.

For $0 < C < 90$, Cos C is positive so: $c^2 < a^2 + b^2$

For C = 90, Cos C = 0 so: $c^2 = a^2 + b^2$

For $90 < C < 180$, Cos C is negative so: $c^2 > a^2 + b^2$

Example: If a = 3, b = 4, C = 120 degrees, we get c2 = 37, so c is a little over 6 compared to 5 for C = 90 degrees.